On certain Properties of Numbers, by the REV. SAMUEL VINCE, *A. M. F. R. S. and Plumian Professor of Astronomy, in the University of Cambridge. An extract of a letter to the Rev. J. Brinkley,* D. D. F. R. S. M. R. I. A. *and Andrews' Professor of Astronomy in the University of Dublin.*

Ramsgate, June 26, 1810.

EULER in his *Introductio in Analysim Infinitorum*, in the chapter *de partitione Numerorum*, has shown, that by a combination of the numbers in each of the Geometric Series 1, 2, 4, 8, &c. and 1, 3, 9, 27, &c. all the natural numbers 1, 2, 3, 4, &c. may be formed, as far as the sum of each series goes. This he has proved, from assuming the products of an indefinite number of factors $(1+x)(1+x^2)(1+x^4)(1+x^8)$ &c. in the first instance, and $(x^{-1}+1+x^1)(x^{-3}+1+x^3)(x^{-9}+1+x^9)$&c. in the second; shewing that in each case, such products may be represented by a series containing the terms $1+x+x^2+x^3+x^4+$ &c. the indices of which must necessarily arise from the combination of the indices in the assumed factors. But

the property now stated may be otherwise proved in a very simple manner immediately from the expression for the sum of each series. I have also added the rules for filling up the intervals of the terms, which Euler has not given; and shewed under what circumstances, other series will have the same property.

First, for the series 1, 2, 4, 8, &c.

The sum of $1+2+4+8+\cdots 2^{n-1}=2^{n}-1=S$; hence, $S+1=2^{n}$ the next term. The difference, then, between the sum S of n terms, and the next term 2^{n}, is 1; therefore the sum S of n terms, carries on within 1 of the next term. If therefore you can for n terms, make up all the natural numbers to their sum, you make them all up to the number next less than the next term 2^{n}; and by adding all those numbers to 2^{n}, you get all the numbers to the number next less than 2^{n+1}. If therefore the rule be true for n terms, it must be true for $n+1$ terms. Now if we take two terms 1, 2, we get $1+2=3$, that is, we get all the numbers as far as the sum of the two numbers, and within 1 of the next term. But, as proved above, if the rule be true for 2 terms, it must be true for 3 terms; if true for 3 terms, it must be true for 4 terms; and so on; hence, the rule is true in general.

Secondly, for the Series 1, 3, 9, 27, &c.

The sum of $1+3+9+27+\cdots\cdots 3^{n-1}=\frac{3^n-1}{2}=S$; hence, $2S+1=3^n$ the next term. The difference, then, between the sum S of n terms and the next term 3^n is $S+1$; therefore the sum S subtracted from the next term 3^n, leaves $S+1$; that is, it brings you back to the number next greater than the sum S. If therefore you can for n terms make up all the numbers to S, the same numbers subtracted from the $\overline{n+1}^{th}$ term will bring you back to $S+1$, the number next greater than S; thus you fill up all the numbers in the interval between the n^{th} term and the $\overline{n+1}^{th}$ term; and if the same numbers be added to the $\overline{n+1}^{th}$ term, you make up all the numbers as far as the sum of $n+1$ terms; if therefore the rule be true for n terms, it must be true for $n+1$ terms. Now if we take two terms 1, 3, we have $3-1=2$, $3+1=4$, and 4 subtracted from the next term 9, leaves 5 the next number greater than the sum of two terms. But, as proved above, if the rule be true for 2 terms, it must be true for 3 terms; if true for 3 terms, it must be true for 4 terms; and so on; hence, the rule is true in general.

The intervals of the *first* series may be filled up by the following Rule.

Let A be any number, and 2^n the term next less than A. Take 2^r next less than $A-2^n$; 2^s next less than $A-2^n-2^r$;

2^t next less than $A-2^n-2^r-2^s$, and so on till there be no remainder; and then $2^n+2^r+2^s+2^t+$ &c. $= A$.

In the *second* series, all the numbers in the general interval from $3^n - 3^{n-1} - 3^{n-2}$ &c. ------ 1 to $3^n+3^{n-1}+3^{n-2}$ + &c. ------ +1, including those terms, may be made up by the following Rule.

After 3^n for the *first* term put -3^{n-1} for 3^{n-1} times, then cyphers as often, and then + 3^{n-1} as often.

For the *second* term put -3^{n-2} for 3^{n-2} times, then cyphers as often, and then $+3^{n-3}$ as often; this to be continued *three* times.

For the *third* term put -3^{n-3} for 3^{n-3} times, then cyphers as often, and then $+3^{n-3}$ as often; this to be continued *nine* times.

For the *fourth* term put -3^{n-4} for 3^{n-4} times, then cyphers as often, and then $+3^{n-4}$ as often; this to be continued *twenty-seven* times.

In general, for the $r^{th.}$ term put -3^{n-r} for 3^{n-r} times, then cyphers as often, and then $+3^{n-r}$ as often; this to be continued 3^{r-1} times.

Proceed thus through all the terms, and you will fill up all the numbers.

But besides these two series, there are many others which have the same property; of these, the two first terms must

necessarily be 1, 2, or 1, 3, or the interval between the two first terms cannot be filled up. The series must also have this further property, that the sum preceding any term (*P*) must reach at least half way from (*P*) to the next term (*Q*), or to (Q—1). The following series have this property.

1, 2, 5, 10, 17, &c.
1, 2, 7 17, 33, &c.
1, 3, 6, 10, 15, &c.
1, 3, 9, 19, 33, &c.

and many others; but the series which requires the smallest number of terms to fill up the interval from 1 to any given number, is, 1, 3, 9, 27, 81, &c.